Stochastic Robustness

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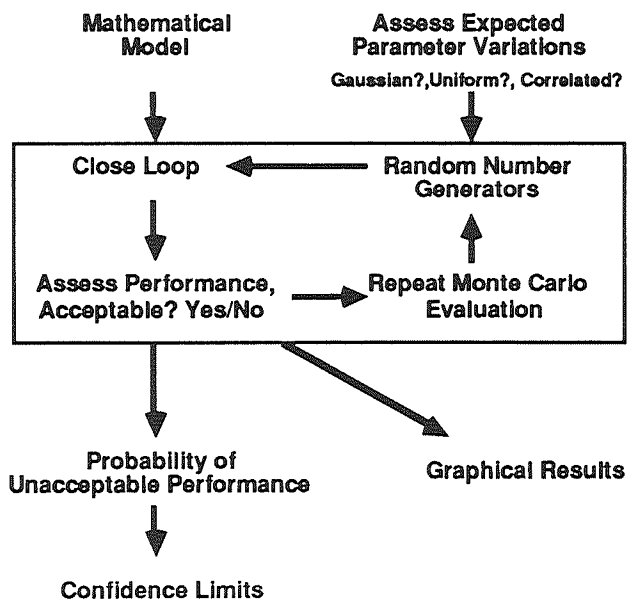
Joint University Program

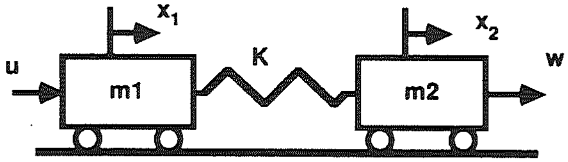
Stochastic robustness analysis (SRA) gives a direct, scalar measure of a conrol system's robustness by assessing the probability that the actual system will have acceptable performance.

Stochastic Robustness Analysis



To carry out stochastic robustness analysis, an expected probability distribution is assigned to each uncertain parameter in the system. The Monte Carlo analysis proceeds by repeatedly assigning shaped random values to each plant parameter, evaluating the stability or performance metric, and performing the binary classification (stable/unstable, etc.). If the system is stable, the state response to a unit disturbance impulse can be propagated to establish whether the response would violate settling time envelopes and whether peak actuator use would violate predetermined maximums. The final estimates of the probability of each form of unacceptable behavior are found by dividing the number of cases in which the overall system had that form of unacceptability by the number of cases run. Stability robustness can be portrayed graphically using the stochastic root locus and by using histograms of parameter values found in the unacceptable cases.

Benchmark Problem



$$y = x2 + v$$
$$z = x2$$

$$Twz = \frac{(1/m_2)(s^2 + k/m_1)}{s^2[s^2 + k(m_1 + m_2)/m_1 m_2]}$$

$$k_o=1, \quad m_1=1, \quad m_2=1$$

Nominal Roots at 0, 0, ±1.41

This benchmark problem was presented at the 1990 American Controls Conference.

The benchmark plant consists of a dual-mass/single-spring system with non-colocated sensor and actuator, as shown in the Fig where x1 and x2 are the positions of the two masses, x3 and x4 are their velocities, and u is a control force on m1. The plant is subject to the disturbance w on m2, and the measurement of x2 is corrupted by noise v.

The baseline plant is undamped, with eigenvalues at ±j(k(m1 + m2)/m1m2), 0, and 0. By the problem specification, a single-input/single-output feedback controller must close its loop around T_{wz}.

Benchmark Design Task

Hard Requirements

1) The closed loop system should be Stable for 0.5 <k <2.

2) The Settling Time after an impulsive disturbance on the nominal system should be less than 15 secs.

Soft Requirements

3) The system should be robust against variations in m_1 and m_2.

4) There should be minimal control usage.

5) The rejection of noise should be good.

Optional

6) The system should compensate for a 0.5 rad/s disturbance input.

Three design problems are posed.
Problem 1 requires a) 15-sec settling time for unit disturbance impulse and nominal mass-spring values (m1 = m2 = k = 1), and b) closed-loop stability for fixed values of mass and 0.5 < k < 2. It is further directed that "reasonable" robustness should be achieved and that controller effort and controller complexity should be minimized. An optional problem replaces the unit disturbance by a sinusoidal disturbance with 0.5-rad frequency. Asymptotic rejection of this signal should be achieved with a 20-sec settling time for the nominal system.

Parameter Variations for Testing

All Probability Distributions Uniform

1) $0.5 < k < 2$

2) $0.5 < k < 2$
   $0.5 < m_1 < 1.5$
   $0.5 < m_2 < 1.5$

3) $0.5 < k < 2$
   $0.5 < m_1 < 1.5$
   $0.5 < m_2 < 1.5$
   $0. < c < 0.1$ (internal damping)
   $0.9 < f < 1.1$ (loop gain variation)
   $0.001 < \tau < 0.4$ sec. (actuator lag)

Closed Loop Transfer Function

$$Twz = \frac{(s^2 m_1 + sc + k)}{(s^4 m_1 m_2 + s^3 c(m_1 + m_2) + s^2 k(m_1 + m_2)) - f(s\tau + 1)(sc + k)(C)}$$

where C is the controller transfer function.

These are the parameter variations used for the Monte Carlo Analysis.

Performances Assessed

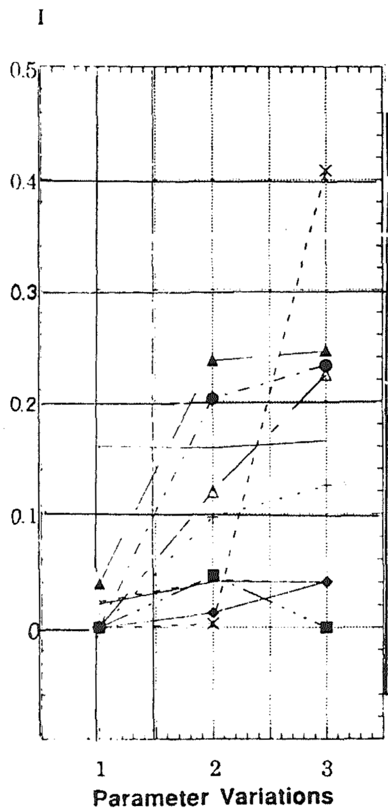
$\mathbb{P}_I$, ***probability of instability.*** **Tested by eigenvalue calculation.**

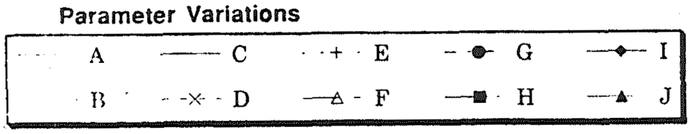
$\mathbb{P}_{Ts0.1}$, ***probability of settling time exceedance.*** **Response to unit w impulse falling outside a ±0.1 - unit envelope >15 secs after impulse**

$\mathbb{P}_{u1}$, ***probability of control-limit exceedance.*** **Using same time histories testing if peak actuator use >1 in response to unit disturbance impulse**

$\mathbb{P}_f$, ***probability of unsatisfactory sinusoidal disturbance rejection.*** **Steady state frequency response at w = 0.5 rad/sec tested for Mag > 0dB.**

Results of Stability Assessment

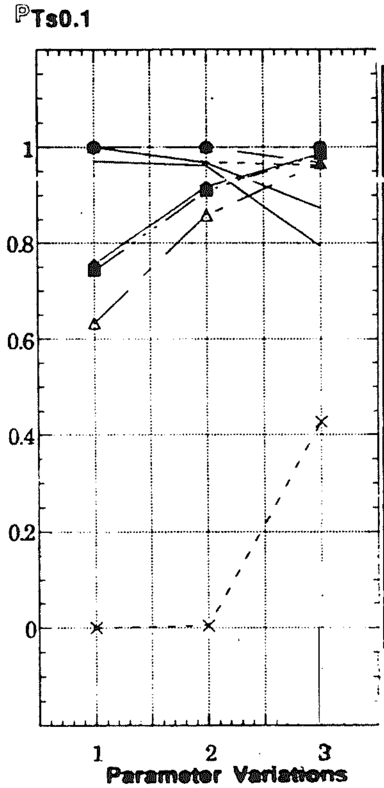




Design	Nominal Values			
	Ts (secs)	Umax (unit)	GM (dB)	PM (deg)
A	21.0	0.514	2.56	26.7
B	19.5	0.469	3.27	26.8
C	19.7	0.468	3.27	26.5
D	9.9	297.8	4.86	58.7
E	21.8	0.900	2.39	22.0
F	13.7	0.732	5.15	23.7
G	21.6	1.458	3.61	25.4
H	18.9	0.574	3.33	24.5
I	22.9	0.416	4.57	27.5
J	43.2	1.047	1.46	63.0

The results of the stability analysis show several interesting features. For instance design D has a large gain and phase margin and a very good nominal settling time; and with parameter variations 1 and 2 the probability of instability is the least of all the designs but when the slight time delay is added in Parameter variation 3 its probability of instability is the worst. Looking at Design I we see a very similar gain and phase margin yet its probability of instability is one of the best under parameter variation 3. This shows that gain and phase margin can be very poor indicators of relative stability.

Results of Settling Time Assessment



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	Nominal Values			
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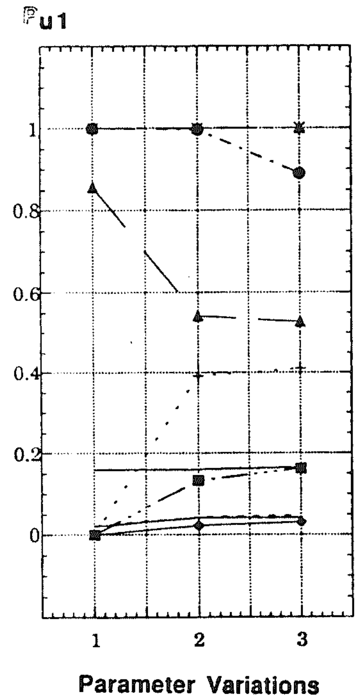
The results of the probability of settling time violation again show us that the nominal times can be misleading. Design F has a nominal settling time of 13.7 secs but under parameter variation 3 there is a higher probability of it violating the 15 sec settling time than Design A which has a nominal settling time of 21 secs.

Results of Control Saturation Assessment



Design	Nominal Values			
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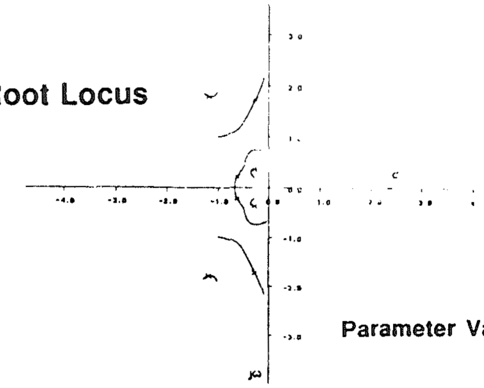
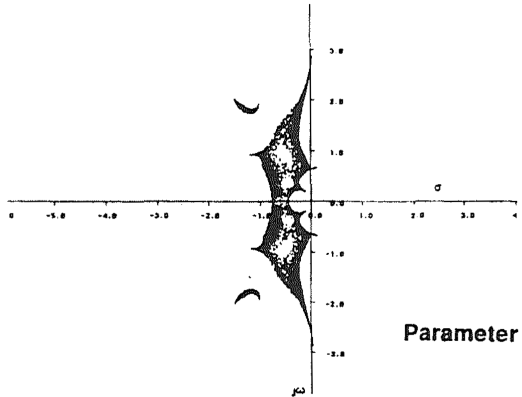
The probabilities of control saturation are reflected by the nominal values of control use.

Stochastic Root Locus

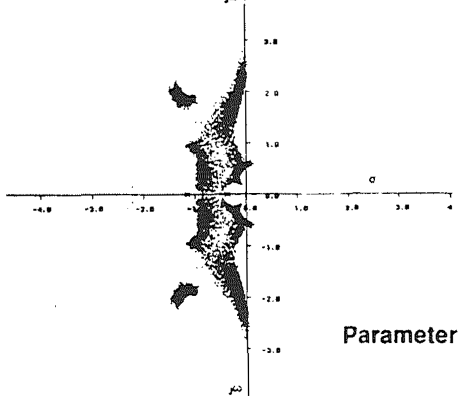
Design H



Parameter Variation 1)



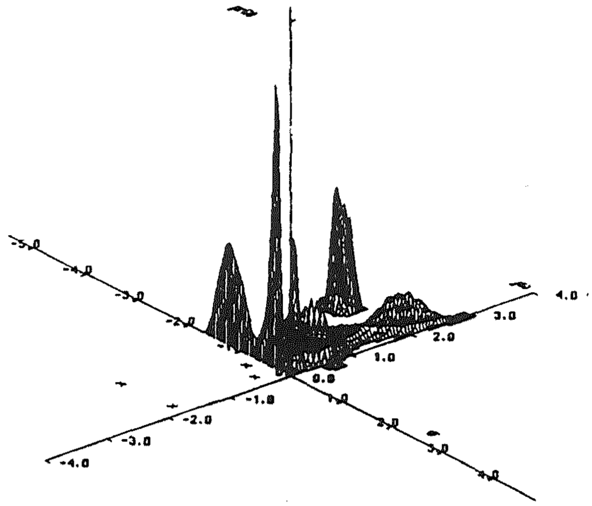
Parameter Variation 2)



Parameter Variation 3)

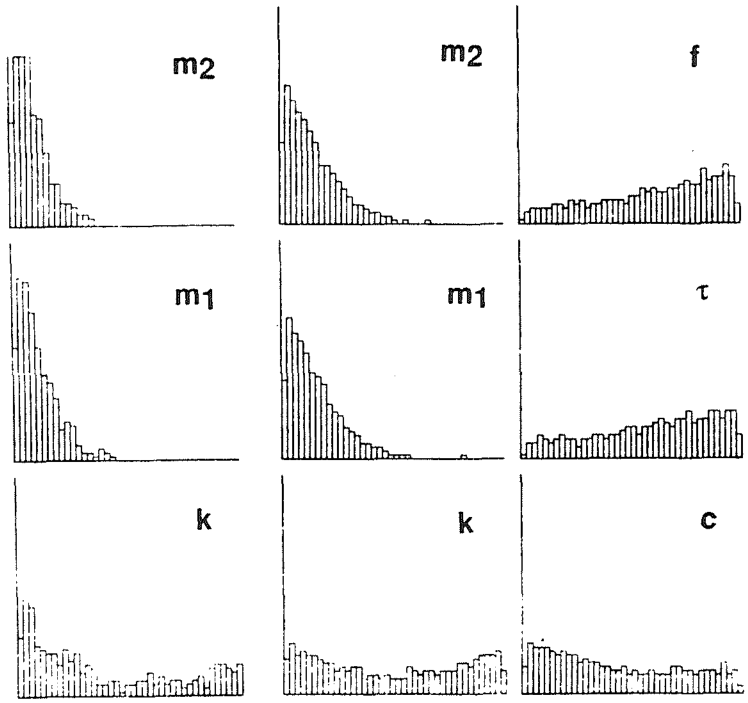
We can obtain graphical data regarding the effects of the parameter variations by looking at the stochastic root locus. For this design we find where the roots will most probably lie, how close the system is to instability and at what frequency unstable roots will have; here we can see two sets of unstable roots, one low frequency, the other high.

Three Dimensional Stochastic Root Locus



The root density can be more clearly seen by plotting in a third dimension.

Design H, Parameter Histograms

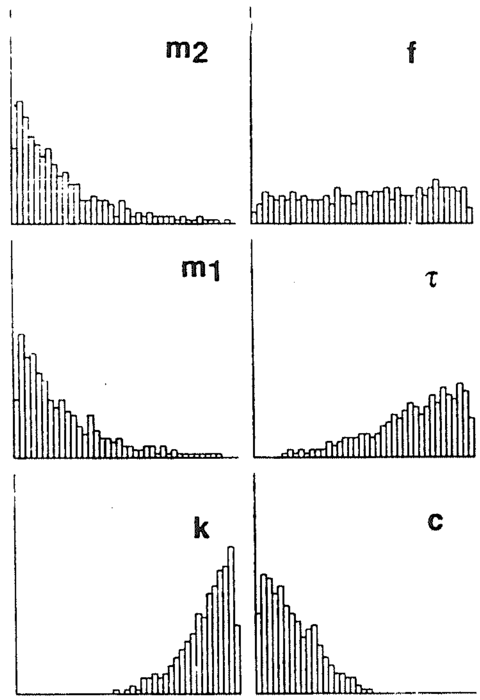


Parameter Variation 2) Parameter Variation 3)

By storing the value of the parameters each time the closed loop is found to be unstable we can see which parameters are causing instability. Here we see that m1 and m2 have the strongest effect. From this graph we could suggest that lower nominal values of m1 and m2 should be used in the synthesis.

Design H, Parameter Histograms

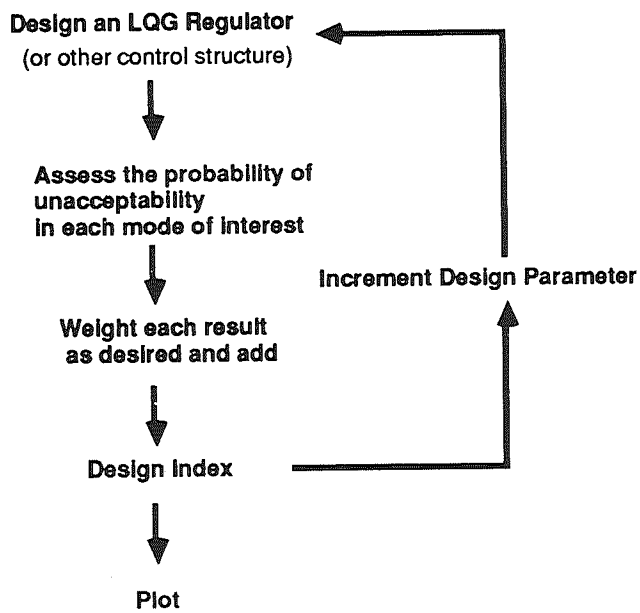
Parameter Variation 3)



High Frequency Roots

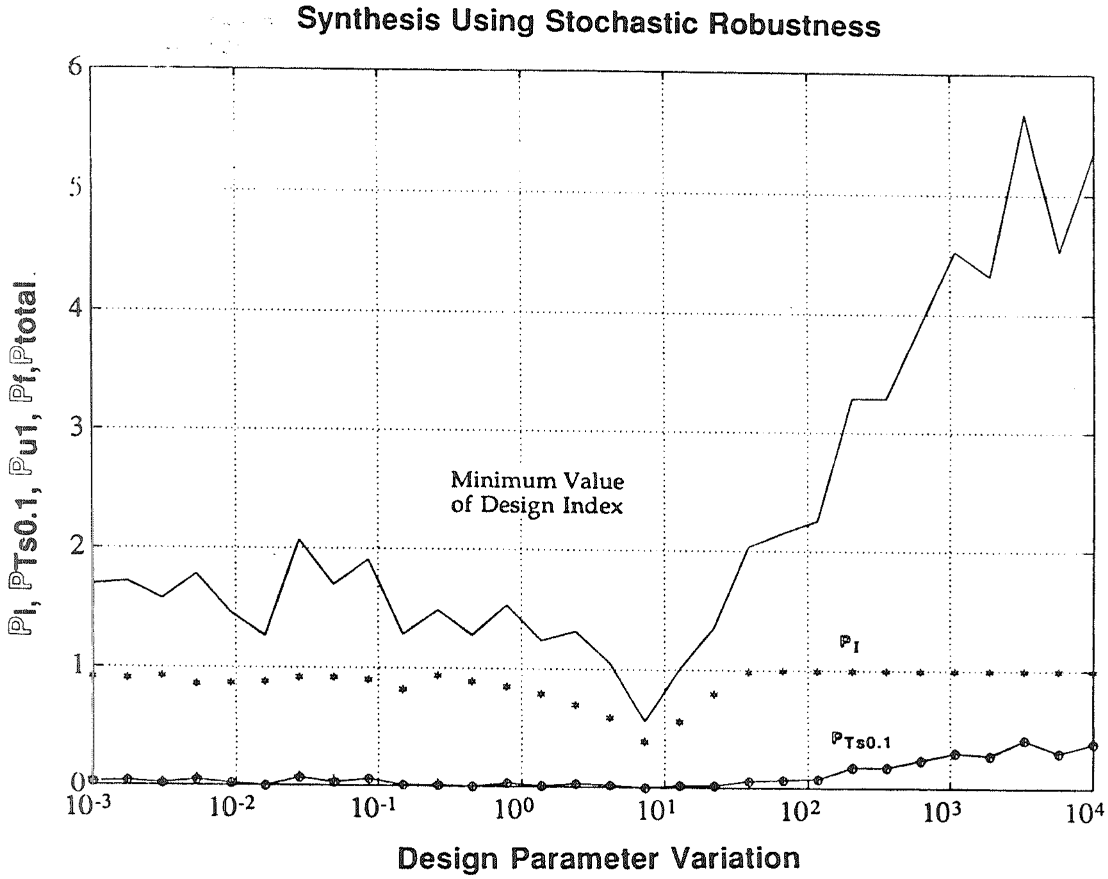
By storing parameters of the high and low frequency roots separately we can determine which parameters cause which instability. For instance here we see that low values of k do not cause high frequency instability.

Synthesis using Stochastic Robustness



Using the fact that Stochastic Robustness Analysis allows us to rank control systems in terms of their overall probability of performing satisfactorily we can carry out synthesis. We first create a series of similar controllers by adjusting a design parameter eg gain, then carry out a Monte Carlo analysis on each one.

Depending on which performances are considered important the results are weighted and added together eg 10*PI+3*Pu+1*PTs. The result is a curve showing the weighted probability of satisfactory performance against the value of the design parameter. There will typically be a minimum in this curve showing the most robust design.



In this rudimentary attempt at synthesis we can see a distinct minimum in the curve of the weighted design index. Setting the design parameter to this value ($10^{0.8}$) we will achieve a good combination of stability and performance robustness. This design seems to be better than the others synthesized for the benchmark problem but further fine tuning and analysis needs to be done.

Conclusions

We can use Stochastic Robustness techniques very flexibly

We can obtain information which is not obvious from other sources

We can rank control systems and suggest changes ie aid synthesis